

First order non-homogeneous q -difference equation for Stieltjes function characterizing q -orthogonal polynomials

J. Arvesú, and A. Soria-Lorente

Department of Mathematics,

Universidad Carlos III de Madrid,

Avda. de la Universidad, 30, 28911, Leganés, Madrid, Spain

March 3, 2013

Abstract

In this paper we give a characterization of some classical q -orthogonal polynomials in terms of a difference property of the associated *Stieltjes function*, i.e this function solves a first order non-homogeneous q -difference equation. The solutions of the aforementioned q -difference equation (given in terms of hypergeometric series) for some canonical cases, namely, q -Charlier, q -Kravchuk, q -Meixner and q -Hahn are worked out.

Keywords: Characterization for orthogonal polynomials, moment functionals, orthogonal polynomials, q -Hypergeometric series, Special functions, Stieltjes function.

2010 Mathematics Subject Classification: Primary 42C05, 33C47, 33E30; Secondary 33D45, 33D15

1 Introduction

The goal of this paper is to extend to the q -discrete orthogonal polynomials on the non-uniform lattice $x(s) = (q^s - 1) / (q - 1)$, the characterization for classical orthogonal polynomials in terms of the standard Stieltjes function (see [23] for original and historical issues on the Stieltjes function; for modern and advanced notions see [8], [15], [25] and the references therein).

The properties of q -Charlier, q -Kravchuk, q -Meixner and q -Hahn orthogonal families are well known (see [2, 20] and [19]): they satisfy a hypergeometric-type difference equation (10); their finite differences constitute an orthogonal polynomial family (16); they can be expressed by a Rodrigues-type formula (15); their associated orthogonalizing weights satisfy a Pearson-type difference equation (13)-(14); they verify a three-term recurrence relation (17). These properties, among others, characterize the classical orthogonal polynomials (see e.g.

[1, 13, 14, 17]). Most of these characterization properties have been already extended to the q -polynomials. However, special attention must be given to [19], particularly for the unified study of some classical q -polynomials carried out. An algebraic approach developed by Maroni [16] is used as a crucial ingredient. Indeed, the starting point is a distributional difference equation fulfilled by the moment functional with respect to which the aforementioned q -polynomials are orthogonal. The motivation for studying the problem considered in the present paper comes precisely from an unproven assertion given in Proposition 2.27 (b) of the aforementioned paper [19]. In this paper we will prove the above mentioned assertion for classical q -polynomials on the lattice $(q^s - 1)/(q - 1)$, i.e., a q -moment functional \mathcal{U} is classic if and only if the Stieltjes function associated with it, given in terms of the q -falling factorial, solves a non-homogeneous version of the Pearson-type difference equation associated with \mathcal{U} (see Theorem 1 below).

The advantages of the proposed approach over existing works consist of working directly with linear functionals rather than power series expansion for the Stieltjes function, i.e., of looking for the intrinsic properties that may satisfy the linear functional under consideration acting, either on the vector space of formal power series or on the vector space of polynomials. Our characterization provides a general framework which can be extended -like in the continuous case- to the study of q -semiclassical orthogonal polynomials [19]. In addition, for illustrating our main result, the Stieltjes function given in terms of the q -falling factorial for the four canonical cases, namely, q -Charlier, q -Kravchuk, q -Meixner, and q -Hahn is explicitly calculated and expressed in terms of hypergeometric functions. As far as we know, there were not explicit hypergeometric expressions for these Stieltjes functions in the literature.

The structure of the paper is as follows. In Section 2, we give a quick overview of the relationship between orthogonal polynomials and Stieltjes function. In Section 3, we compress some necessary definitions and tools. Lastly, in Section 4 the Stieltjes function for q -orthogonal polynomials in terms of the q -falling factorial basis is introduced and the main theorem for the proposed characterization is proved. This result constitutes a q -analogue of the characterization given in [15] for classical discrete orthogonal polynomials (see also [9]). Finally, in Section 5 the explicit expressions for the corresponding solutions of the difference equation that characterizes the classical q -orthogonal polynomials on the non-uniform lattice $x(s) = (q^s - 1)/(q - 1)$, i.e. the Stieltjes functions, are studied.

2 Stieltjes function and orthogonal polynomials

We start by reviewing some results needed for the sequel; mainly extracted from [8, 20, 22]. The linear space of polynomials with coefficients in \mathbb{C} (the set of complex numbers) is denoted by \mathbb{P} , and its dual space by \mathbb{P}^* , whose elements are linear functionals (moment functionals).

Let the action of $\mathcal{U} \in \mathbb{P}^*$ on $p \in \mathbb{P}$ be denoted by the duality bracket $\langle \mathcal{U}, p \rangle$.

The moments of \mathcal{U} are given by $\langle \mathcal{U}, x^k \rangle = u_k$, $k = 0, 1, \dots$, and by linearity yields the following relation $\langle \mathcal{U}, P_n(x) \rangle = \sum_{k=0}^n \alpha_k u_k$, where $P_n(z) = \sum_{k=0}^n \alpha_k z^k$.

Formally, if \mathcal{U} is a moment functional determined by the moment sequence $\{u_k\}_{k \geq 0}$, the Stieltjes function associated with \mathcal{U} can be defined as the generating function of these moments by means of the series expansion

$$S(z) = \sum_{k \geq 0} \frac{u_k}{z^{k+1}}. \quad (1)$$

Notice that $S(z)$ corresponds to its sum when the Laurent expansion (1) converges for some z ; otherwise $S(z)$ represents its analytic continuation -provided that it exists-.

The above function (1) can be approximated by rational functions with prescribed order near infinity [22]. This approximation problem -often known as $(n-1, n)$ Padé approximation near infinity- consists in finding the polynomials $P_n(z) = \sum_{k=0}^n \alpha_k z^k$ and $Q_{n-1}(z) = \sum_{k=0}^{n-1} \beta_k z^k$, such that

$$P_n(z)S(z) - Q_{n-1}(z) = \mathcal{O}(z^{-n-1}), \quad (2)$$

where $\deg P_n \leq n$ and $\deg Q_{n-1} \leq n-1$.

By getting $Q_{n-1}(z)$ equal to the polynomial part of $P_n(z)S(z)$ and using the prescribed order at infinity in the interpolation condition (2), the following linear system of n homogeneous equations [22]

$$\sum_{i=0}^n u_{i+j} \alpha_i = 0, \quad j = 0, \dots, n-1, \quad (3)$$

for determining the $n+1$ unknown coefficients of $P_n(z)$, holds. Hence, $P_n(z)$ is determined up to a multiplicative factor. Observe that the linear system (3) is equivalent to the following orthogonality conditions:

$$\langle \mathcal{U}, P_n(x)x^k \rangle = 0, \quad k = 0, \dots, n-1. \quad (4)$$

Consequently, $P_n(z)$ is orthogonal with respect to any polynomial of degree less than n for the linear functional \mathcal{U} . On the other hand, we have selected $Q_{n-1}(z)$ equals to the polynomial part of $P_n(z)S(z)$, hence one has

$$Q_{n-1}(z) = \left\langle \mathcal{U}, \frac{P_n(z) - P_n(x)}{z - x} \right\rangle, \quad (5)$$

where it is assumed that \mathcal{U} acts on the variable x . Notice that (5) leads to the formal relation

$$P_n(z) \left\langle \mathcal{U}, \frac{1}{(z-x)} \right\rangle - Q_{n-1}(z) = \left\langle \mathcal{U}, \frac{P_n(x)}{z-x} \right\rangle, \quad (6)$$

which indeed implies -at least formal- the linearity extension of the linear functional \mathcal{U} to the vector space of formal power series. Moreover, regardless of the

convergence of (1), it is straightforward to see that this series provides a means of approximating $\langle \mathcal{U}, (z-x)^{-1} \rangle$. Indeed, for every nonnegative integer n ,

$$\left\langle \mathcal{U}, \frac{1}{z-x} \right\rangle = \left\langle \mathcal{U}, \sum_{k=0}^{n-1} \frac{x^k}{z^{k+1}} \right\rangle + \left\langle \mathcal{U}, \left(\frac{x}{z}\right)^n \frac{1}{z-x} \right\rangle.$$

Thus, for $z \rightarrow \infty$, in every sector $\epsilon < \arg z < \pi - \epsilon$ ($0 < \epsilon < \pi/2$) one gets $|z-x| \geq |z| \sin \epsilon$. Hence,

$$\left\langle \mathcal{U}, \frac{1}{z-x} \right\rangle - \sum_{k=0}^{n-1} \frac{u_k}{z^{k+1}} = \mathcal{O}_\epsilon(z^{-n-1}),$$

which holds for all z in the above sector. Analogously, for the remainder approximation term we get the prescribed order z^{-n-1} near ∞ , in the above sector, i.e

$$\left\langle \mathcal{U}, \frac{P_n(x)}{z-x} \right\rangle = \mathcal{O}_\epsilon(z^{-n-1}).$$

Moreover, we recall that in the framework of the algebraic approach developed by Maroni [16, 17, 18] it is established a topological isomorphism between the space \mathbb{P}^* and the space of formal power series endowed with appropriate topologies. Indeed, it is always possible to associate $(P_n)_{n \geq 0}$ of degree n with a unique sequence $(\mathcal{U}_n)_{n \geq 0}$, $\mathcal{U}_n \in \mathbb{P}^*$, called the dual sequence of $(P_n)_{n \geq 0}$, such that $\langle \mathcal{U}_n, P_m \rangle = 0$, for $n \neq m$, and equals 1 for $n = m$. Indeed, any functional $\mathcal{U} \in \mathbb{P}^*$ can be represented as follows

$$\mathcal{U} = \sum_{n \geq 0} \langle \mathcal{U}, P_n \rangle \mathcal{U}_n,$$

which makes use of the linearly extension of \mathcal{U} to the vector space of formal power series. Therefore, the Stieltjes function associated with \mathcal{U} in the Padé approximation problem (2) has the formal representation

$$S(z) = \langle \mathcal{U}, (z-x)^{-1} \rangle. \quad (7)$$

From now on we will consider a specific class of moment functionals. Let μ be a Borel measure on the real line \mathbb{R} (with infinitely many points of increase), supported on $\Omega \subset \mathbb{R}$. We define the linear functional on which we will focus our attention

$$\langle \mathcal{U}, p \rangle = \int_{\Omega} p(x) d\mu(x), \quad p \in \mathbb{P}. \quad (8)$$

In particular, we will consider

$$\mu = \sum_{k=0}^N \rho(x) \delta_{x_k}, \quad \rho(x_k) > 0, \quad x_k \in \mathbb{R}, \quad N \in \mathbb{N} \cup \{+\infty\}.$$

This measure μ is discrete (with finite moments $u_k = \int_{\Omega} x^k d\mu(x)$, $k \geq 0$) formed by a linear combination of Dirac measures at the points x_0, \dots, x_N . By

\mathbb{N} we denotes the set of all nonnegative integers. Therefore, the orthogonality condition (4) for the q -polynomials on the lattice $\{x(s) \mapsto \mathbb{R}^+ : s = 0, \dots, N\}$ with respect to (8) is defined as follows –see [24, eq. (4.7), p.256] as well as [21]

$$\langle \mathcal{U}, P_n x^k \rangle = \sum_{s=0}^N P_n(x(s)) x^k(s) \rho(s) \Delta x(s - 1/2) = 0, \quad k = 0, \dots, n-1, \quad (9)$$

where, $\Delta x(s) = x(s+1) - x(s)$ denotes the forward difference operator. In general, the polynomial $P_n(x(s))$ is called discrete orthogonal polynomial. In what follows we will denote any polynomial $P_n(x(s))$ simply as $P_n(s)$.

The term ‘*classical discrete orthogonal polynomial*’ [20] (also known as classical orthogonal polynomials of a discrete variable) obeys the fact that $x(s) = c_1 q^s + c_2 q^{-s} + c_3$, ($q \in \mathbb{R}^+ \setminus \{1\}$) or $x(s) = c_4 s^2 + c_5 s + c_6$, where the constants $c_i \in \mathbb{R}$ ($i = 1, \dots, 6$) are independents of variable s , and the orthogonalizing weights $\rho(s)$ are solutions of the Pearson-type difference equation (see formulas (13)-(14) below and [20, pp. 70-72]).

A remarkable feature of the classical discrete orthogonal polynomials is that they satisfy the hypergeometric-type difference equation

$$\begin{aligned} \sigma(s) \frac{\Delta}{\Delta x(s - 1/2)} \frac{\nabla y(s)}{\nabla x(s)} + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda y(s) &= 0, \\ \sigma(s) = \tilde{\sigma}(x(s)) - (\tilde{\tau}(x(s)) \Delta x(s - 1/2))/2, \quad \tau(s) &= \tilde{\tau}(x(s)), \end{aligned} \quad (10)$$

where $\nabla y(s) = \Delta y(s - 1)$ is the backward difference operator, and

$$\tilde{\sigma}(s) = a_2 x^2(s) + a_1 x(s) + a_0 \quad (11)$$

$$\tilde{\tau}(s) = b_1 x(s) + b_0. \quad (12)$$

Equation (10) is a discretization of the very well known hypergeometric differential equation (see [20])

$$\tilde{\sigma}(x) y''(x) + \tilde{\tau}(x) y'(x) + \tilde{\lambda} y(x) = 0, \quad \text{with } \deg \tilde{\sigma} \leq 2, \quad \deg \tilde{\tau} = 1, \quad \tilde{\lambda} \in \mathbb{R}.$$

Notice that the solution $\rho(s)$ of the Pearson-type equation (written in two equivalent forms)

$$\Delta [\sigma(s) \rho(s)] = \tau(s) \rho(s), \quad \Delta \stackrel{\text{def}}{=} \frac{\Delta}{\Delta x(s - 1/2)}, \quad (13)$$

$$\nabla [(\sigma(s) + \tau(s) \nabla x(s + 1/2)) \rho(s)] = \tau(s) \rho(s), \quad \nabla \stackrel{\text{def}}{=} \frac{\nabla}{\nabla x(s + 1/2)}, \quad (14)$$

allows to write (10) in self-adjoint form. This symmetrization factor of (10) with the condition $\sigma(s) \rho(s) x^k(s - 1/2)|_{s=0, N+1} = 0$, $k = 0, 1, \dots$, is called the orthogonalizing weight of (9). This boundary condition is compatible with the existence of all moments of (9).

It is well-known that many properties of orthogonal polynomials can be deduced from (9); in particular, equation (10). In turn from this equation

follow important properties like the Rodrigues-type formula (see [20, 24] for general theory and [2] for main data of q -polynomials on non-uniform lattice $x(s) = (q^s - 1)/(q - 1)$).

$$P_n(s) = \frac{B_n}{\rho(s)} \nabla^{(n)}[\rho_n(s)], \quad \nabla^{(n)} = \frac{\nabla}{\nabla x_1(s)} \frac{\nabla}{\nabla x_2(s)} \cdots \frac{\nabla}{\nabla x_n(s)}, \quad (15)$$

where $x_k(s) = x(s + \frac{k}{2})$, and $\rho_n(s) = \rho(s + n) \prod_{m=1}^n \sigma(s + m)$; the orthogonality of the differences

$$\sum_{s=0}^{N-1} \frac{\Delta P_n(s)}{\Delta x(s)} \frac{\Delta P_m(s)}{\Delta x(s)} \rho_1(s) \Delta x(s) = \delta_{n,m} \left\| \frac{\Delta P_n(s)}{\Delta x(s)} \right\|^2, \quad (16)$$

where $\|\cdot\|$ denotes the norm in the Hilbert space L_μ^2 ; the three-term recurrence relation

$$x(s)P_n(s) = \alpha_n P_{n+1}(s) + \beta_n P_n(s) + \gamma_n P_{n-1}(s), \quad n = 0, 1, \dots, N, \quad (17)$$

where $P_{-1} = 0$, P_0 is a real constant, and the coefficients α_n , β_n , and γ_n are given explicitly in terms of (11) and (12) (see [2], expressions in (42)).

A second linearly independent solution of equation (10) –for a specific choice $\lambda = \lambda_n$, see [20]– is the so-called function of the second kind on non-uniform lattice (see [24], formula (5.1)). This function is also a second linearly independent solution of equation (17). For a relationship between the function of the second kind and the Stieltjes function we refer to [20, 24].

3 Basic definitions and notations

Here, we will use the q -analogue of the Pochhammer symbol [11, 12, 20]

$$(a; q)_k = \prod_{0 \leq j \leq k-1} (1 - aq^j), \quad \text{for } k > 0, \quad \text{and } (a; q)_0 = 1. \quad (18)$$

By Pochhammer symbol $(s)_k$ and falling factorial $[s]_k$ we mean

$$(s)_k = s(s+1) \cdots (s+k-1), \quad (s)_0 = 1, \quad k \geq 1, \\ [s]_k = (-1)^k (-s)_k,$$

respectively. Notice that the subscript in the above expressions is a nonnegative integer, whereas in the sequel symbol $[s]_q$ denotes the q -number defined as follows

$$[s]_q = \frac{q^{s/2} - q^{-s/2}}{q^{1/2} - q^{-1/2}}, \quad s \in \mathbb{C}.$$

For the q -factorial we use the following definition

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q, \quad n \in \mathbb{N}, \quad [0]_q! = 1.$$

Observe that in [11, p. 7] the q -number is defined in a different way.

The q -hypergeometric series ${}_r\varphi_s$ is defined as

$${}_r\varphi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right) = \sum_{k \geq 0} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} \frac{z^k}{(q; q)_k},$$

where $\binom{m}{n}$ denotes the binomial coefficient.

In what follows, the q -falling factorial defined as

$$[s]_q^{(k)} = \prod_{0 \leq j \leq k-1} x(s-j), \quad \text{for } k > 0, \quad \text{and} \quad [s]_q^{(0)} = 1, \quad (19)$$

where $x(s) = \frac{q^s - 1}{q - 1}$, will play the same role that the power x^k plays in a formal series expansion. Indeed, the q -falling factorial basis $\{1, [s]_q^{(1)}, \dots, [s]_q^{(n)}, \dots\}$ has been suggested to be more natural than the basis $\{1, x(s), \dots, x^n(s), \dots\}$, mainly when multiple orthogonal polynomials are considered [5]. Observe that the q -falling factorial $[s]_q^{(k)}$ is a polynomial of degree k in $x(s)$. Accordingly, the action on it of the forward and backward difference operators defined in (13)-(14) gives

$$\Delta [s]_q^{(k)} = q^{3/2-k} [k]_q^{(1)} [s]_q^{(k-1)}, \quad \text{and} \quad \nabla [s]_q^{(k)} = q^{1/2-k} [k]_q^{(1)} [s-1]_q^{(k-1)},$$

respectively.

In addition, the q -falling factorial can be rewritten in terms of the q -analogue of the Pochhammer symbol as follows

$$[s]_q^{(k)} = \frac{(q^{-s}; q)_k}{(q-1)^k} q^{k(s - \frac{k-1}{2})}, \quad k \geq 1, \quad \text{and} \quad [s]_q^{(0)} = 1. \quad (20)$$

Clearly, the Pochhammer symbol and the falling factorial can be recovered as a limiting case

$$\lim_{q \rightarrow 1} [s]_q^{(k)} = (-1)^k (-s)_k = [s]_k.$$

Let \mathbb{S} be the spanning set of $\{1, [s]_q^{(1)}, \dots, [s]_q^{(n)}, \dots\}$ over \mathbb{R} . Observe that \mathbb{S} coincides with the linear space of polynomials of discrete variable $x(s)$ with real coefficients. \mathbb{S}_n denotes the corresponding linear subspace of dimension n . Let us define the algebraic dual space of \mathbb{S} , denoted by \mathbb{S}^* , as

$$\mathbb{S}^* = \{\mathcal{U} : \mathbb{S} \rightarrow \mathbb{R}, \text{ such that } \mathcal{U} \text{ is a linear functional}\}.$$

The real number $u_n^q = \langle \mathcal{U}, [s]_q^{(n)} \rangle$, $n \geq 0$, is said to be the q -moment of order n of \mathcal{U} and the sequence $\{u_n^q\}_{n \geq 0}$ is called the q -moment sequence associated with \mathcal{U} .

Definition 1 Let $\{u_n^q\}_{n \geq 0}$ be a sequence in \mathbb{R} . \mathcal{U} is said to be the q -moment functional associated with the sequence $\{u_n^q\}_{n \geq 0}$, if $\langle \mathcal{U}, [s]_q^{(n)} \rangle = u_n^q$ and it can be extended to \mathbb{S} by linearity, i.e., if

$$r(s) = \sum_{k=0}^n a_k [s]_q^{(k)}, \text{ then } \langle \mathcal{U}, r \rangle = \sum_{k=0}^n a_k u_k^q.$$

Accordingly, from Definition 1 and equation (9) one has

$$\langle \mathcal{U}, r \rangle = \sum_{s=0}^N r(s) \rho(s) \triangle x(s - \tfrac{1}{2}), \quad (21)$$

where, the q -moment of order k is given by

$$u_k^q = \langle \mathcal{U}, [s]_q^{(k)} \rangle = \sum_{s=0}^N [s]_q^{(k)} \rho(s) \triangle x(s - \tfrac{1}{2}), \quad k = 0, 1, \dots \quad (22)$$

Below we sketch some definitions and results that we will use in the next sections (for comprehensive definitions and theorems on q -moment functionals we refer to [19]).

Any moment functional \mathcal{U} , which maps \mathbb{S} into \mathbb{S} has a transpose $\mathcal{V}^T : \mathbb{S}^* \mapsto \mathbb{S}^*$ defined by $\langle \mathcal{V}^T \mathcal{U}, p \rangle = \langle \mathcal{U}, \mathcal{V} p \rangle$, $\forall p \in \mathbb{S}$, $\forall \mathcal{U} \in \mathbb{S}^*$; then $\Delta \mathcal{U}$, $\nabla \mathcal{U}$, and $p\mathcal{U}$ are defined by duality according to the following definitions.

Definition 2 Let \mathcal{U} be a q -moment functional. The forward and backward differences of the moment functional \mathcal{U} , denoted by $\Delta \mathcal{U}$, and $\nabla \mathcal{U}$, respectively, are defined as follows

$$\begin{aligned} \langle \Delta \mathcal{U}, p(s) \rangle &= -\langle \mathcal{U}, \nabla p(s) \rangle, \quad \text{where } p \in \mathbb{S}, \\ \langle \nabla \mathcal{U}, p(s) \rangle &= -\langle \mathcal{U}, \Delta p(s) \rangle. \end{aligned}$$

Here, the forward difference operator Δ on moment functionals is minus the transpose of the backward difference operator on polynomials.

Definition 3 Let $p \in \mathbb{S}$. The linear functional $\mathcal{V} = p\mathcal{U}$ is said to be the left-multiplication of \mathcal{U} by a polynomial p if

$$\langle \mathcal{V}, r \rangle = \langle \mathcal{U}, pr \rangle, \quad r \in \mathbb{S}.$$

Definition 4 If (σ, τ) are polynomials of minimum degree such that

$$\Delta(\sigma \mathcal{U}) = \tau \mathcal{U}, \quad (23)$$

define the class of \mathcal{U} as the nonnegative integer number s such that

$$s = \max\{\deg \sigma - 2, \deg \tau - 1\}.$$

The polynomial sequence $\{P_n(s)\}_{n \geq 0}$ orthogonal with respect to the q -moment functional \mathcal{U} of class zero is said to be classical orthogonal polynomial sequence.

Equation (23) is a q -analogue of the functional Pearson equation [10, 15]. In general, the polynomial sequences orthogonal with respect to such moment functionals \mathcal{U} are said to be semiclassical orthogonal polynomial sequences.

4 Stieltjes function for q -orthogonal polynomials in terms of the q -falling factorials

Assume that \mathcal{U} is the moment functional determined by the q -moment sequence $\{u_k^q\}_{k \geq 0}$. Below, in the same fashion that the continuous case (1) we will define a q -analogue of the Stieltjes function associated with \mathcal{U} (in short q -Stieltjes function) as the generating function of these q -moments by means of a series expansion, but using the q -falling factorial basis (19) instead. More precisely,

Definition 5 *Let \mathcal{U} be a q -moment functional defined on the linear space \mathbb{S} . The q -Stieltjes function associated with \mathcal{U} is defined as the formal series expansion*

$$S_q(z) = \sum_{k \geq 0} \frac{u_k^q}{q^k [z]_q^{(k+1)}}. \quad (24)$$

The following result deals with the q -analogue of the formal representation (7). Indeed, we will use a q -analogue of the Chu-Vandermonde convolution (see [12] for details)

$${}_2\varphi_1 \left(\begin{matrix} q^{-n}, b \\ c \end{matrix} \middle| q; q^n c/b \right) = \frac{(cb^{-1}; q)_n}{(c; q)_n}.$$

In particular, one can select $s \in \{0, 1, \dots, N\}$ and $z \in \mathbb{C}$ such that

$${}_2\varphi_1 \left(\begin{matrix} q^{-s}, q \\ q^{1-z} \end{matrix} \middle| q; q^{s-z} \right) = \frac{(q^{-z}; q)_s}{(q^{1-z}; q)_s}. \quad (25)$$

We will use this expression in the proof of the following lemma as well as the linearity extension of \mathcal{U} to the linear space of formal power series in accordance with the discussion given in section 2 with the usual adaptation required for discrete measures. Indeed, using the relation (see p. 262 [24])

$$\frac{1}{x(s) - x(z)} = -\frac{1}{x(z)} \sum_{k=0}^{n-1} \left(\frac{x(s)}{x(z)} \right)^k + \frac{x^n(s)}{[x(s) - x(z)]x^n(z)},$$

and proceeding like in (6) one gets an asymptotic expansion of (30) as $x(z) \rightarrow \infty$. However, we refer to [16, 17, 18] for a more comprehensive analysis.

Lemma 1 *The q -Stieltjes function (24) admits the following formal representation*

$$S_q(z) = \left\langle \mathcal{U}, [x(z) - x(s)]^{-1} \right\rangle, \quad (26)$$

where it is assumed that \mathcal{U} acts on the discrete variable $x(s)$.

Proof. Relations (20) and (22) enable us to rewrite (24) as follows

$$S_q(z) = \sum_{k \geq 0} \frac{\langle \mathcal{U}, [s]_q^{(k)} \rangle}{q^k [z]_q^{(k+1)}} = \sum_{k \geq 0} \frac{(q-1) \langle \mathcal{U}, (q^{-s}; q)_k q^{ks - \binom{k}{2}} \rangle}{q^k (q^{-z}; q)_{k+1} q^{(k+1)z - \binom{k+1}{2}}}. \quad (27)$$

Particularizing the well known relation $(a; q)_{k+n} = (a; q)_n (aq^n; q)_k$ -see [11]- for the choice $a = q^{-z}$ and $n = 1$, i.e.

$$(q^{-z}; q)_{k+1} = (q^{-z}; q)_1 (q^{1-z}; q)_k,$$

it follows that equation (27) can be rewritten as

$$\begin{aligned} S_q(z) &= \frac{q-1}{q^z-1} \sum_{k \geq 0} \frac{\langle \mathcal{U}, q^{k(s-z)} (q^{-s}; q)_k \rangle}{(q^{1-z}; q)_k} \\ &= \frac{1}{x(z)} \left\langle \mathcal{U}, \sum_{k \geq 0} \frac{(q^{-s}; q)_k (q; q)_k}{(q^{1-z}; q)_k (q; q)_k} \left(\frac{q^s q^{1-z}}{q} \right)^k \right\rangle. \end{aligned} \quad (28)$$

If one uses (25) equation (28) transforms into the expression

$$S_q(z) = \left\langle \mathcal{U}, \frac{1}{x(z)} \frac{(q^{-z}; q)_s}{(q^{1-z}; q)_s} \right\rangle. \quad (29)$$

Taking into account relation (18), a straightforward calculation leads to the following equalities

$$\frac{1}{x(z)} \frac{(q^{-z}; q)_s}{(q^{1-z}; q)_s} = \frac{(1-q)(1-q^{-z})}{(1-q^z)(1-q^{-z+s})} = \frac{1}{x(z) - x(s)},$$

which makes expression (29) coincident with (26); therefore the statement holds.

Remark 1 From (21) and (26) one can write (24) up to a constant factor as

$$S_q(z) = \sum_{s=0}^N \frac{\rho(s) \triangle x(s-1/2)}{x(z) - x(s)}, \quad (30)$$

which reveals its singularities.

The Stieltjes function (30) is related to the function of the second kind on non-uniform lattice (see [24], formula (5.1)) for the specific choice $n = 0$. This function of the second kind is the second linearly independent solution of equations (10) and (17), respectively.

In the next results the subscripts are used to indicate on which variable the difference operators are acting.

Lemma 2 The following relation

$$\left\langle \nabla_s \mathcal{U}, \frac{1}{x(z) - x(s)} \right\rangle = \left\langle \mathcal{U}, \nabla_z \frac{1}{x(z) - x(s)} \right\rangle, \quad (31)$$

holds.

Proof. By using the operator ∇ defined in (14) one can compute the backward difference of the Stieltjes function given in the q -falling factorial basis. Indeed, by Lemma 1 one gets

$$\begin{aligned}\nabla S_q(z) &= \frac{1}{\nabla x(z + \frac{1}{2})} [S_q(z) - S_q(z - 1)] \\ &= \frac{1}{\nabla x(z + \frac{1}{2})} \left[\left\langle \mathcal{U}, \frac{1}{x(z) - x(s)} \right\rangle - \left\langle \mathcal{U}, \frac{1}{x(z - 1) - x(s)} \right\rangle \right],\end{aligned}$$

or equivalently,

$$\nabla S_q(z) = \left\langle \mathcal{U}, \nabla_z \frac{1}{x(z) - x(s)} \right\rangle.$$

A straightforward computation leads to the following relation

$$\nabla_z \frac{1}{x(s) - x(z)} = \frac{q^{1/2}}{\pi(s, z)} = \Delta_s \frac{1}{x(z) - x(s)},$$

where

$$\begin{aligned}\pi(s, z) &= [x(s + 1) - x(z)] [x(s) - x(z)] \\ &= q [x(s) - x(z - 1)] [x(s) - x(z)],\end{aligned}\tag{32}$$

is a polynomial of degree two both in the variable s and z .

Hence,

$$\nabla S_q(z) = - \left\langle \mathcal{U}, \Delta_s \frac{1}{x(z) - x(s)} \right\rangle,$$

which implies that equation (31) is fulfilled.

The following theorem establishes the main result in this paper.

Theorem 1 *The polynomial sequence $\{P_n(s)\}_{n \geq 0}$ orthogonal with respect to the moment functional \mathcal{U} is classical if and only if the Stieltjes function (24) (in terms of q -falling factorials) satisfies the following first order non-homogeneous difference equation*

$$\nabla [(\sigma(s) + \tau(s) \nabla x(s + 1/2)) S_q(s)] = \tau(s) S_q(s) + C_q, \quad C_q \in \mathbb{R} \setminus \{0\}, \tag{33}$$

where the constant

$$C_q = \left(a_2 q^{-1/2} + \frac{1}{2} b_1 q^{-1} (q - 1) - b_1 \right) u_0^q,$$

depends on the polynomial coefficients a_2 and b_1 of $\tilde{\sigma}(s)$ and $\tilde{\tau}(s)$, respectively (see formulas (11)-(12)).

Proof. Let us introduce the linear functional $\mathcal{V} = p(s)\mathcal{U}$, where

$$\begin{aligned}p(s) &= \sigma(s) + \tau(s) \nabla x(s + \frac{1}{2}) \\ &= \alpha x^2(s) + \beta x(s) + \gamma,\end{aligned}$$

and

$$\begin{aligned}\alpha &= a_2 + \frac{1}{2}b_1q^{-1/2}(q-1), \\ \beta &= a_1 + \frac{1}{2}b_1q^{-1/2} + \frac{1}{2}b_0q^{-1/2}(q-1), \\ \gamma &= a_0 + \frac{1}{2}b_0q^{-1/2}.\end{aligned}$$

Here we have used that

$$\begin{aligned}\nabla x(s + \tfrac{1}{2}) &= q^{-1/2}(q-1) \left[x(s) + \frac{1}{q-1} \right] \\ &= q^{-1/2}(q-1)x(s) + q^{-1/2}.\end{aligned}$$

Notice that

$$\frac{x(s) - x(s-1)}{x(s+1/2) - x(s-1/2)} = q^{-1/2}.$$

Hence,

$$\begin{aligned}\nabla p(s) &= \alpha \nabla x^2(s) + \beta \nabla x(s) \\ &= \alpha \left[\frac{x^2(s) - x^2(s-1)}{x(s+1/2) - x(s-1/2)} \right] + \beta \left[\frac{x(s) - x(s-1)}{x(s+1/2) - x(s-1/2)} \right] \\ &= \alpha q^{-1/2} [x(s) + x(s-1)] + \beta q^{-1/2}.\end{aligned}\tag{34}$$

Thus,

$$\nabla p(s) = \alpha q^{-1/2} [x(s) + x(s-1)] + \beta q^{-1/2}.$$

Now, if one assumes that \mathcal{U} is classic (see equation (23)), or equivalently,

$$\nabla [p(s)\mathcal{U}] = \tau(s)\mathcal{U},\tag{35}$$

one gets

$$\left\langle \nabla [p(s)\mathcal{U}], \frac{1}{x(z) - x(s)} \right\rangle = \left\langle \tau(s)\mathcal{U}, \frac{1}{x(z) - x(s)} \right\rangle.\tag{36}$$

According to Definition 3, one gets for the right hand-side of the equation (36) the following relation

$$\begin{aligned}\left\langle \tau(s)\mathcal{U}, \frac{1}{x(z) - x(s)} \right\rangle &= \left\langle \mathcal{U}, \frac{\tau(s)}{x(z) - x(s)} \right\rangle = \left\langle \mathcal{U}, \frac{\tau(s) - \tau(z) + \tau(z)}{x(z) - x(s)} \right\rangle \\ &= \tau(z)S_q(z) - b_1 \langle \mathcal{U}, 1 \rangle = \tau(z)S_q(z) - b_1 u_0^q.\end{aligned}\tag{37}$$

By Lemma 2 and taking into account the formula before (32), we have

$$\left\langle \nabla_s \mathcal{V}, \frac{1}{x(z) - x(s)} \right\rangle = - \left\langle \mathcal{V}, \frac{q^{1/2}}{\pi(s, z)} \right\rangle.$$

Therefore, from Definition 3, for the left-hand side of equation (36) one gets

$$\left\langle \nabla_s [p(s) \mathcal{U}], \frac{1}{x(z) - x(s)} \right\rangle = -q^{1/2} \left\langle \mathcal{U}, \frac{p(s)}{\pi(s, z)} \right\rangle.$$

Thus,

$$\begin{aligned} \left\langle \mathcal{U}, \frac{p(s)}{\pi(s, z)} \right\rangle &= \left\langle \mathcal{U}, \frac{p(s) - p(z-1) + p(z-1)}{\pi(s, z)} \right\rangle \\ &= -q^{-1/2} p(z-1) \nabla S_q(z) + \left\langle \mathcal{U}, \frac{p(s) - p(z-1)}{\pi(s, z)} \right\rangle. \end{aligned}$$

Since,

$$p(s) - p(z-1) = \alpha [x^2(s) - x^2(z-1)] + \beta [x(s) - x(z-1)].$$

From expression (32) we have

$$\frac{p(s) - p(z-1)}{\pi(s, z)} = \frac{\alpha}{q} \frac{x(s) + x(z-1)}{x(s) - x(z)} + \frac{\beta}{q} \frac{1}{x(s) - x(z)}.$$

Thus,

$$\begin{aligned} \left\langle \mathcal{U}, \frac{p(s) - p(z-1)}{\pi(s, z)} \right\rangle &= \frac{\alpha}{q} \left\langle \mathcal{U}, \frac{x(s) + x(z-1)}{x(s) - x(z)} \right\rangle - \frac{\beta}{q} \left\langle \mathcal{U}, \frac{1}{x(z) - x(s)} \right\rangle \\ &= \frac{\alpha}{q} u_0^q - \left(\frac{\alpha}{q} [x(z) + x(z-1)] + \frac{\beta}{q} \right) S_q(z). \end{aligned}$$

Taking into account (34) the above expression transforms into the equation

$$\left\langle \mathcal{U}, \frac{p(s) - p(z-1)}{\pi(s, z)} \right\rangle = \alpha q^{-1} u_0^q - q^{-1/2} S_q(z) \nabla p(z).$$

Therefore, the following expression for the left-hand side of (36)

$$\begin{aligned} \left\langle \nabla [p(s) \mathcal{U}], \frac{1}{x(z) - x(s)} \right\rangle &= p(z-1) \nabla S_q(z) + S_q(z) \nabla p(z) - \alpha q^{-1/2} u_0^q \\ &= \nabla [p(z) S_q(z)] - \alpha q^{-1/2} u_0^q. \end{aligned} \quad (38)$$

holds.

Then, by using (36), (37), and (38) one gets

$$\begin{aligned} \nabla [p(z) S_q(z)] &= \tau(z) S_q(z) + \left(\alpha q^{-1/2} - b_1 \right) u_0^q \\ &= \tau(z) S_q(z) + \left[a_2 q^{-1/2} + \frac{1}{2} b_1 q^{-1} (q-1) - b_1 \right] u_0^q, \end{aligned}$$

and equation (33) holds.

Notice that the statement holds since the converse implication, i.e. (33) \implies (35), follows by the above chain of equalities but proceeding in reverse order.

Remark 2 *Theorem 1 can be proved in a more elementary way, i.e., without using the assertions of Lemma 1 and Lemma 2, respectively. However, this procedure requires some cumbersome calculations.*

Below we highlight the main ideas of this procedure. Assume that \mathcal{U} verifies (23), or equivalently

$$\left\langle \nabla [(\sigma(s) + \tau(s) \nabla x(s + 1/2))\mathcal{U}], [s]_q^{(k)} \right\rangle = \left\langle \tau(s)\mathcal{U}, [s]_q^{(k)} \right\rangle.$$

Hence,

$$\left\langle \mathcal{U}, [\sigma(s) + \tau(s) \nabla x(s + 1/2)] q^{1-k/2} [k]_q [s]_q^{(k-1)} + \tau(s) [s]_q^{(k)} \right\rangle = 0. \quad (39)$$

From equation (39), after some straightforward computations, one gets a three-term recurrence relation involved the following q -moments u_{k-1}^q , u_k^q , and u_{k+1}^q . The resulting equation can be expressed in the following convenient way

$$\Gamma_k + \Psi_k = \Xi_k, \quad (40)$$

where

$$\begin{aligned} \Gamma_k = & - \left[a_2 q^{\frac{3k}{2}-1} + \frac{1}{2} b_1 (q-1) q^{\frac{3k-3}{2}} \right] [k+2]_q u_{k+1}^q - [k+1]_q \left[2a_2 q^{k-1} [k]_q \right. \\ & - a_2 q^{\frac{3k-3}{2}} + a_1 q^{\frac{k-1}{2}} + \frac{1}{2} b_1 q^{\frac{k}{2}-1} (q^{k-1} + q^k - 1) + \frac{1}{2} b_0 q^{\frac{k}{2}-1} (q-1) \left. \right] u_k^q \\ & - [k]_q \left[a_2 q^{\frac{k}{2}-1} [k-1]_q^2 + (a_1 + \frac{1}{2} b_1 q^{k-\frac{3}{2}}) [k-1]_q + (a_0 + \frac{1}{2} b_0 q^{k-\frac{3}{2}}) q^{1-\frac{k}{2}} \right] u_{k-1}^q, \\ \Psi_k = & \left[a_2 q^{k-\frac{3}{2}} (q+1) + \frac{1}{2} b_1 q^{k-2} (q^2 - 1) \right] u_{k+1}^q + \left\{ a_2 \left[q^{\frac{k}{2}-2} (q+1) [k]_q - q^{-\frac{3}{2}} \right] \right. \\ & \left. + a_1 q^{-\frac{1}{2}} + \frac{1}{2} b_1 \left[q^{\frac{k-1}{2}} (1 - q^{-2}) [k]_q + q^{-2} \right] + \frac{1}{2} \frac{(q-1)}{q} b_0 \right\} u_k^q, \end{aligned}$$

and

$$\Xi_k = b_1 q^k u_{k+1}^q + (b_1 q^{\frac{k-1}{2}} [k]_q + b_0) u_k^q.$$

Dividing Γ_k , Ψ_k , and Ξ_k by $q^k [s]_q^{(k+1)}$ and summing from $k = 0$ to ∞ , after tedious calculations, one obtains

$$\begin{aligned} \sum_{k \geq 0} \frac{\Gamma_k}{q^k [s]_q^{(k+1)}} = & \left[a_2 q^{-\frac{3}{2}} + \frac{1}{2} b_1 q^{-2} (q-1) \right] u_0^q \\ & + [\sigma(s-1) + \tau(s-1) \nabla x(s-1/2)] \nabla S_q(s), \end{aligned} \quad (41)$$

$$\begin{aligned} \sum_{k \geq 0} \frac{\Psi_k}{q^k [s]_q^{(k+1)}} = & - \left[a_2 q^{-\frac{3}{2}} (q+1) + \frac{1}{2} b_1 (1 - q^{-2}) \right] u_0^q \\ & + \nabla [\sigma(s) + \tau(s) \nabla x(s+1/2)] S_q(s), \end{aligned} \quad (42)$$

and

$$\sum_{k \geq 0} \frac{\Xi_k}{q^k [s]_q^{(k+1)}} = -b_1 u_0^q + \tau(s) S_q(s). \quad (43)$$

Recall that from (40) one has

$$\sum_{k \geq 0} \frac{\Gamma_k}{q^k [s]_q^{(k+1)}} + \sum_{k \geq 0} \frac{\Psi_k}{q^k [s]_q^{(k+1)}} = \sum_{k \geq 0} \frac{\Xi_k}{q^k [s]_q^{(k+1)}}.$$

Then, relations (41)-(43) yields the equation (33). The converse implication follows by the chain of obtained equalities but proceeding in reverse order.

Observe that the more standard properties that characterize the q -classical orthogonal polynomials can be derived from a distributional difference equation (35). For this observation we refer to the approach given in [19] in the framework of a pure algebraic approach –used also in the above proof. Accordingly, from Theorem 1 follows that the q -classical orthogonal polynomials verify the hypergeometric-type difference equation (10), the three-term recurrence relation (17); their finite differences constitute an orthogonal polynomial family (16) and they can be expressed by a Rodrigues-type formula (15), among other aforementioned properties.

5 Examples

In this Section, we will give an explicit expression -case by case- for the moment sequence (22) as well as for the solution of (33) in terms of q -hypergeometric series, i.e. of expression (24).

5.1 q -Charlier case

The q -Charlier polynomials are orthogonal with respect to the q -moment functional (21) defined by the weight function (see e.g. [2, eq. (87)])

$$\rho(s) = \frac{\mu^s}{e_q[(1-q)\mu] \Gamma_q(s+1)}, \quad \mu > 0, \quad 0 < (1-q)\mu < 1,$$

where $s \in [0, \infty)$, and $e_q(z)$ denotes the q -analogue of the exponential function (for details, see [11]).

Observe that the above function $\rho(s)$ is a solution of equation (14) for the polynomial coefficients: $\tau(s) = \mu q^{3/2} - q^{1/2} x(s)$, $\sigma(s) = q^s x(s)$, and

$$\sigma(s) + \tau(s) \nabla x(s + \tfrac{1}{2}) = \mu q^{s+1}.$$

These polynomial coefficients (see e.g. [2, eq. (86)]) are involved in the characteristic equation (33) as well as the moment u_0^q . For determining this moment we now compute explicitly all the moments of (21) for the above choice of $\rho(s)$.

According to (22) one has

$$\begin{aligned} u_k^q &= \frac{q^{-1/2}}{e_q[(1-q)\mu]} \sum_{s \geq k} [s]_q^{(k)} \frac{\tilde{\mu}^s}{\Gamma_q(s+1)} = \frac{q^{-1/2}}{e_q[(1-q)\mu]} \sum_{s \geq k} \frac{\tilde{\mu}^s}{\frac{(q; q)_{s-k}}{(1-q)^{s-k}}} \\ &= \frac{q^{-1/2} \tilde{\mu}^k}{e_q[(1-q)\mu]} \sum_{n \geq 0} \frac{\tilde{\mu}^n}{\frac{(q; q)_n}{(1-q)^n}}, \quad \text{where } \tilde{\mu} = q\mu. \end{aligned}$$

Thus, the explicit expression for the k -th moment associated with (21) is as follows

$$u_k^q = \frac{\tilde{\mu}^k e_q[(1-q)\tilde{\mu}]}{q^{1/2} e_q[(1-q)\mu]}, \quad k = 0, 1, \dots$$

In particular

$$u_0^q = \frac{e_q[(1-q)\tilde{\mu}]}{q^{1/2} e_q[(1-q)\mu]}.$$

Finally, from (24) one gets

$$S_q(z) = \frac{u_0^q}{x(z)} \sum_{k \geq 0} \frac{(q; q)_k (-1)^k q^{\binom{k}{2}} [\mu(1-q)q^{1-z}]^k}{(q^{1-z}; q)_k (q; q)_k},$$

since the q -falling factorial can be rewritten as follows (see (20))

$$[z]_q^{(k+1)} = (-1)^k \frac{x(z)}{(1-q)^k} (q^{1-z}; q)_k q^{(z-1)k - \binom{k}{2}}. \quad (44)$$

Hence, the q -Stieltjes function associated with the q -Charlier moment functional that solves equation (33) is given in terms of the following hypergeometric series

$$S_q(z) = \frac{u_0^q}{x(z)} {}_1\varphi_1 \left(\begin{matrix} q \\ q^{1-z} \end{matrix} \middle| q; \mu(1-q)q^{1-z} \right). \quad (45)$$

On the other hand, we know that (45) must be equal to (30). This fact is easily established if one rewrites $\rho(s)$ as

$$\rho(s) = C_q \frac{[(1-q)\mu]^s}{(q; q)_s}, \quad \text{where } C_q = \frac{1}{e_q[(1-q)\mu]},$$

and (30) as

$$\begin{aligned} S_q(z) &= \frac{C_q q^{-1/2}}{x(z)} \sum_{s \geq 0} \frac{(q^{-z}; q)_s [(1-q)\mu q]^s}{(q^{1-z}; q)_s (q; q)_s} \\ &= \frac{C_q q^{-1/2}}{x(z)} \sum_{s \geq 0} \frac{(q^{-z}; q)_s (0; q)_s [(1-q)\mu q]^s}{(q^{1-z}; q)_s (q; q)_s}. \end{aligned}$$

Here we have used that $\Gamma_q(s+1) = \frac{(q;q)_s}{(1-q)^s}$.

Thus, the q -Stieltjes function associated with the q -Charlier moment functional also has the form

$$S_q(z) = \frac{C_q q^{-1/2}}{x(z)} {}_2\varphi_1 \left(\begin{matrix} q^{-z}, 0 \\ q^{1-z} \end{matrix} \middle| q; (1-q)\mu q \right).$$

Now, using the Heine's transformation formula (for details, see [12] page 16)

$${}_2\varphi_1 \left(\begin{matrix} a, 0 \\ c \end{matrix} \middle| q; z \right) = e_q[z] {}_1\varphi_1 \left(\begin{matrix} a^{-1}c \\ c \end{matrix} \middle| q; az \right), \quad a \neq 0, |z| < 1,$$

under the assumption $|(1-q)\mu q| < 1$ one gets the desired equality:

$$\begin{aligned} S_q(z) &= \frac{C_q q^{-1/2}}{x(z)} {}_2\varphi_1 \left(\begin{matrix} q^{-z}, 0 \\ q^{1-z} \end{matrix} \middle| q; (1-q)\mu q \right) \\ &= \frac{u_0^q}{x(z)} {}_1\varphi_1 \left(\begin{matrix} q \\ q^{1-z} \end{matrix} \middle| q; \mu(1-q)q^{1-z} \right). \end{aligned}$$

5.2 q -Kravchuk case

The q -Kravchuk polynomials are orthogonal with respect to the q -moment functional (21), where the orthogonalizing weight function

$$\rho(s) = q^{\binom{s}{2}} \frac{[N]_q!}{\Gamma_q(s+1)\Gamma_q(N-s+1)} p^s (1-p)^{N-s}, \quad 0 < p < 1, \quad N \in \mathbb{N},$$

is a solution of Pearson-type equation (14) for the choice (see [4] p.89)

$$\begin{aligned} \tau(s) &= \frac{q^{1/2} p q (q^N - 1)}{1-p} - \frac{q^{1/2} (p(q-1) + 1)}{1-p} x(s) \\ \sigma(s) &= (q-1)x^2(s) + x(s). \end{aligned}$$

By using the expression $\Gamma_q(s+1) = q^{\binom{s}{2}/2} [s]_q!$, the above function $\rho(s)$ can be rewritten as

$$\rho(s) = q^{s(N-1)/2 - \binom{N}{2}/2} \frac{[N]_q!}{[s]_q! [N-s]_q!} p^s (1-p)^{N-s}.$$

Hence, for the moments (22) one gets

$$\begin{aligned}
u_k^q &= q^{-\frac{1}{2}[(\binom{N}{2})+1]} \sum_{s=k}^N q^{s(N+1)/2} [s]_q^{(k)} \frac{[N]_q!}{[s]_q! [N-s]_q!} p^s (1-p)^{N-s} \\
&= q^{-\frac{1}{2}[(\binom{k+1}{2})+(\binom{N}{2})+1]} \sum_{s=k}^N q^{s(N+k+1)/2} \frac{[N]_q!}{[s-k]_q! [N-s]_q!} p^s (1-p)^{N-s} \\
&= q^{\frac{1}{2}[2(\binom{k+1}{2})-(\binom{N}{2})-1]} [N]_q^{(k)} p^k \sum_{n=0}^{N-k} \frac{q^{n(N+k+1)/2} [N-k]_q!}{[n]_q! [N-k-n]_q!} p^n (1-p)^{N-n-k},
\end{aligned}$$

where we have used the following relations

$$\begin{aligned}
\frac{[s]_q^{(k)}}{[s]_q!} &= \frac{q^{ks/2 - (\binom{k+1}{2})/2}}{[s-k]_q!}, \\
[N]_q! &= q^{\frac{1}{2}[(\binom{N-k}{2})-(\binom{N}{2})]} [N]_q^{(k)} [N-k]_q!.
\end{aligned}$$

Equivalently,

$$u_k^q = q^{\frac{1}{2}[2(\binom{k+1}{2})-(\binom{N}{2})-1]} [N]_q^{(k)} p^k \sum_{n=0}^{N-k} q^{n(2k+n+1)/2} \left[\begin{matrix} N-k \\ n \end{matrix} \right]_q p^n (1-p)^{N-n-k},$$

where the q -binomial symbol is defined as (see e.g. [11, p. 24])

$$\left[\begin{matrix} n \\ j \end{matrix} \right]_q = \frac{\Gamma_q(n+1)}{\Gamma_q(k+1) \Gamma_q(n-k+1)} = q^{k(n-k)/2} \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Using the well-known relation (see [6, pp. 3–6])

$$\sum_{j=0}^n q^{\binom{j+1}{2}} \left[\begin{matrix} n \\ j \end{matrix} \right]_q x^{n-j} y^j = \prod_{j=0}^{n-1} (x + yq^{j+1}),$$

the above expression for the k -th moment transforms into

$$u_k^q = q^{\frac{1}{2}[2(\binom{k+1}{2})-(\binom{N}{2})-1]} [N]_q^{(k)} p^k \prod_{j=0}^{N-k-1} (1-p + pq^{j+k+1}).$$

Finally

$$u_k^q = \frac{u_0^q}{(1-q)^k} \frac{(q^{-N}; q)_k}{\left(\frac{pq}{p-1}; q\right)_k} \left(\frac{pq^{N+1}}{p-1}\right)^k,$$

where

$$u_0^q = \frac{(1-p)^N}{\sqrt{q^{\binom{N}{2}+1}}} \left(\frac{pq}{p-1}; q\right)_N.$$

Therefore, from (24) and (44) one gets the following explicit expression for the q -Stieltjes function associated with the q -Kravchuk moment functional

$$S_q(z) = \frac{u_0^q}{x(z)} \sum_{k \geq 0} \frac{(q^{-N}; q)_k (q; q)_k (-1)^k q^{\binom{k}{2}} \left(\frac{pq^{N+1-z}}{p-1} \right)^k}{(q^{1-z}; q)_k \left(\frac{pq}{p-1}; q \right)_k}.$$

This expression can be written in terms of hypergeometric series as follows

$$S_q(z) = \frac{u_0^q}{x(z)} {}_2\varphi_2 \left(\begin{matrix} q^{-N}, q \\ q^{1-z}, \frac{pq}{p-1} \end{matrix} \middle| q; \frac{pq^{N+1-z}}{p-1} \right). \quad (46)$$

Previously, we have established a relationship between (46) and (30). Now, aimed to check a similar relationship between (46) and (30) we rewrite $\rho(s)$ as

$$\rho(s) = C_q \frac{(q^{-N}; q)_s}{(q; q)_s} \left(\frac{pq^N}{p-1} \right)^s, \quad (47)$$

where

$$C_q = (1-p)^N \frac{[N]_q!}{\Gamma_q(N+1)} = (1-p)^N q^{-\binom{N}{2}/2}.$$

In (47) we have taken into account the following relation

$$\frac{\Gamma_q(N-s+1)}{\Gamma_q(N+1)} = \frac{(-1)^s (1-q)^s q^{\binom{s}{2}-Ns}}{(q^{-N}; q)_s}.$$

Thus, from (30) one gets

$$S_q(z) = \frac{C_q q^{-1/2}}{x(z)} \sum_{s \geq 0} \frac{(q^{-N}; q)_s (q^{-z}; q)_s \left(\frac{pq^{N+1}}{p-1} \right)^s}{(q^{1-z}; q)_s (q; q)_s}.$$

Accordingly, the q -Stieltjes function associated with the q -Kravchuk moment functional also has the form

$$S_q(z) = \frac{C_q q^{-1/2}}{x(z)} {}_2\varphi_1 \left(\begin{matrix} q^{-N}, q^{-z} \\ q^{1-z} \end{matrix} \middle| q; \frac{pq^{N+1}}{p-1} \right).$$

Now, using the Jackson's transformation formula (for details, see [12] p. 15)

$${}_2\varphi_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| q; z \right) = \frac{(az; q)_\infty}{(z; q)_\infty} {}_2\varphi_2 \left(\begin{matrix} a, b^{-1}c \\ c, az \end{matrix} \middle| q; bz \right), \quad b \neq 0, \quad (48)$$

one obtains the aforementioned relationship

$$\begin{aligned}
S_q(z) &= \frac{C_q q^{-1/2}}{x(z)} {}_2\varphi_1 \left(\begin{matrix} q^{-N}, q^{-z} \\ q^{1-z} \end{matrix} \middle| q; \frac{pq^{N+1}}{p-1} \right) \\
&= \frac{C_q q^{-1/2}}{x(z)} \frac{\left(\frac{pq}{p-1}; q\right)_\infty}{\left(\frac{pq}{p-1} q^N; q\right)_\infty} {}_2\varphi_2 \left(\begin{matrix} q^{-N}, q \\ q^{1-z}, \frac{pq}{p-1} \end{matrix} \middle| q; \frac{pq^{N+1-z}}{p-1} \right) \\
&= \frac{u_0^q}{x(z)} {}_2\varphi_2 \left(\begin{matrix} q^{-N}, q \\ q^{1-z}, \frac{pq}{p-1} \end{matrix} \middle| q; \frac{pq^{N+1-z}}{p-1} \right).
\end{aligned}$$

5.3 q -Meixner case

The q -Meixner polynomials are orthogonal with respect to the q -moment functional (21) defined by the weight function (see e.g. [2, p. 314])

$$\rho(s) = \frac{\mu^s \Gamma_q(\gamma + s)}{\Gamma_q(\gamma) \Gamma_q(s + 1)} = \frac{\mu^s}{q^{(s)/2} [s]_q!} \frac{(q^\gamma; q)_s}{(1 - q)^s}, \quad 0 < \mu < 1, \quad \gamma > 0. \quad (49)$$

Recall that $\rho(s)$ is a solution of Pearson-type difference equation for the polynomial coefficients (see e.g. [2, eq. (68)], up to the factor $q - 1$):

$$\begin{aligned}
\tau(s) &= q^{1/2} (\mu q^{\gamma+1} - 1) x(s) + \mu q^{\frac{\gamma+2}{2}} [\gamma]_q, \\
\sigma(s) &= (q - 1) x(s)^2 + x(s).
\end{aligned}$$

For the above orthogonalizing weight function and (21), let us compute the associated q -moments (22)

$$\begin{aligned}
u_k^q &= q^{-1/2} \sum_{s \geq 0} q^{s - (s)/2} [s]_q^{(k)} \frac{\mu^s}{[s]_q!} \frac{(q^\gamma; q)_s}{(1 - q)^s} \\
&= q^{-\frac{1}{2}[(\frac{k+1}{2})+1]} \sum_{s \geq k} q^{s(k+2)/2 - (s)/2} \frac{\mu^s}{[s - k]_q!} \frac{(q^\gamma; q)_s}{(1 - q)^s} \\
&= q^{-\frac{1}{2}[(\frac{k+1}{2})+1-k(k+2)]} \mu^k \sum_{n \geq 0} q^{n(k+2)/2 - (n+k)/2} \frac{\mu^n}{[n]_q!} \frac{(q^\gamma; q)_{n+k}}{(1 - q)^{n+k}}.
\end{aligned}$$

Considering the following elementary relations

$$\begin{aligned}
n(k+2) - \binom{n+k}{2} &= n(5-n)/2 - \binom{k}{2}, \\
(q^\gamma; q)_{n+k} &= (q^\gamma; q)_k (q^{\gamma+k}; q)_n,
\end{aligned}$$

one gets

$$\begin{aligned} u_k^q &= q^{k-1/2} \frac{(q^\gamma; q)_k}{(1-q)^k} \mu^k \sum_{n \geq 0} q^{n(5-n)/4} \frac{\mu^n}{[n]_q!} \frac{(q^{\gamma+k}; q)_n}{(1-q)^n} \\ &= q^{k-1/2} \frac{\Gamma_q(\gamma+k)}{\Gamma_q(\gamma)} \mu^k \sum_{n \geq 0} q^{n(5-n)/4} \frac{\mu^n}{[n]_q!} \frac{(q^{\gamma+k}; q)_n}{(1-q)^n}. \end{aligned}$$

Since $[n]_q! = q^{-\binom{n}{2}/2} \frac{(q; q)_n}{(1-q)^n}$, the following relation

$$u_k^q = q^{-1/2} \frac{\Gamma_q(\gamma+k)}{\Gamma_q(\gamma)} \tilde{\mu}^k \sum_{n \geq 0} \frac{\tilde{\mu}^n}{(q; q)_n} (q^{\gamma+k}; q)_n, \quad \text{where } \tilde{\mu} = q\mu,$$

holds. Thus, by the q -binomial theorem (see e.g. [11, eq. 1.3.2]),

$$u_k^q = q^{-1/2} \frac{(\mu q^{\gamma+1}; q)_\infty}{(\mu q; q)_\infty} \frac{\Gamma_q(\gamma+k)}{\Gamma_q(\gamma)} \frac{\tilde{\mu}^k}{(\mu q^{\gamma+1}; q)_k}.$$

In particular

$$u_0^q = q^{-1/2} \frac{(\mu q^{\gamma+1}; q)_\infty}{(\mu q; q)_\infty}.$$

In order to obtain an expression for the corresponding q -Stieltjes function associated with the q -Meixner moment functional one uses the following elementary relation

$$\frac{\Gamma_q(\gamma+k)}{\Gamma_q(\gamma)} = \frac{(q^\gamma; q)_k}{(1-q)^k}, \quad (50)$$

as well as (24) and (44), respectively. Thus,

$$S_q(z) = \frac{u_0^q}{x(z)} \sum_{k \geq 0} \frac{(q^\gamma; q)_k (q; q)_k (-1)^k q^{\binom{k}{2}} (\mu q^{1-z})^k}{(q^{1-z}; q)_k (\mu q^{\gamma+1}; q)_k (q; q)_k}.$$

Equivalently, in terms of hypergeometric series

$$S_q(z) = \frac{u_0^q}{x(z)} {}_2\varphi_2 \left(\begin{matrix} q^\gamma, q \\ q^{1-z}, \mu q^{\gamma+1} \end{matrix} \middle| q; \mu q^{1-z} \right). \quad (51)$$

Again, to establish an explicit relationship between (51) and (30) we rewrite (49) –based on equation (50)– as

$$\rho(s) = \mu^s \frac{(q^\gamma; q)_s}{(q; q)_s}.$$

Thus, from (30) one obtains the following q -Stieltjes function associated with the q -Meixner moment functional

$$S_q(z) = \frac{q^{-1/2}}{x(z)} \sum_{s \geq 0} \frac{(q^\gamma; q)_s (q^{-z}; q)_s (\mu q)^s}{(q^{1-z}; q)_s (q; q)_s},$$

or equivalently, in terms of hypergeometric series

$$S_q(z) = \frac{q^{-1/2}}{x(z)} {}_2\varphi_1 \left(\begin{matrix} q^\gamma; q^{-z} \\ q^{1-z} \end{matrix} \middle| q; \mu q \right).$$

Again, using the Jackson's transformation formula (48) we have the following relation

$$\begin{aligned} S_q(z) &= \frac{q^{-1/2}}{x(z)} {}_2\varphi_1 \left(\begin{matrix} q^\gamma; q^{-z} \\ q^{1-z} \end{matrix} \middle| q; \mu q \right) \\ &= \frac{q^{-1/2}}{x(z)} \frac{(\mu q^{\gamma+1}; q)_\infty}{(\mu q; q)_\infty} {}_2\varphi_2 \left(\begin{matrix} q^\gamma, q \\ q^{1-z}, \mu q^{\gamma+1} \end{matrix} \middle| q; \mu q^{1-z} \right) \\ &= \frac{u_0^q}{x(z)} {}_2\varphi_2 \left(\begin{matrix} q^\gamma, q \\ q^{1-z}, \mu q^{\gamma+1} \end{matrix} \middle| q; \mu q^{1-z} \right). \end{aligned}$$

5.4 q -Hahn case

The q -Hahn polynomials are orthogonal with respect to the q -moment functional (21) defined by the orthogonalizing weight in the interval $[0, N-1]$ (see [3, Table 4.1])

$$\rho(s) = q^{\left(\frac{\alpha+\beta}{2}\right)s} \frac{\tilde{\Gamma}_q(s+\beta+1)\tilde{\Gamma}_q(N+\alpha-s)}{\tilde{\Gamma}_q(s+1)\tilde{\Gamma}_q(N-s)}, \quad \alpha, \beta > -1, \quad N \in \mathbb{N},$$

where $\tilde{\Gamma}(s) = q^{-(s-1)(s-2)/4} f(s; q)$ if $0 < q < 1$, or $\tilde{\Gamma}(s) = f(s; q^{-1})$ if $q > 1$, being $f(s; q) = (1-q)^{1-s} \prod_{k \geq 0} (1-q^{k+1}) / \prod_{k \geq 0} (1-q^{s+k})$. The above function $\rho(s)$ is a solution of equation (14) for the polynomial coefficients (see [3]):

$$\begin{aligned} \sigma(s) &= -q^{-(N+\alpha)/2} x^2(s) + q^{-1/2} [N+\alpha]_q x(s), \\ \tau(s) &= -q^{-(\beta+2-N)/2} [\alpha+\beta+2]_q x(s) + q^{\alpha+\beta+1} [\beta+1]_q [N-1]_q. \end{aligned}$$

For calculating the q -moments (22) we use

$$\begin{aligned} u_k^q &= q^{-1/2} \sum_{s=k}^{N-1} q^{\left(\frac{\alpha+\beta+2}{2}\right)s} [s]_q^{(k)} \frac{\tilde{\Gamma}_q(s+\beta+1)\tilde{\Gamma}_q(N+\alpha-s)}{[s]_q! [N-1-s]_q!} \\ &= q^{-\frac{1}{2}[(\frac{k+1}{2})+1]} \sum_{s=k}^{N-1} q^{\left(\frac{\alpha+\beta+k+2}{2}\right)s} \frac{\tilde{\Gamma}_q(s+\beta+1)\tilde{\Gamma}_q(N+\alpha-s)}{[s-k]_q! [N-1-s]_q!}. \end{aligned}$$

Hence,

$$u_k^q = q^{\chi(k, \alpha, \beta)} \sum_{n=0}^{N-k-1} q^{n\left(\frac{\alpha+\beta+k+2}{2}\right)} \frac{\tilde{\Gamma}_q(\beta+k+1+n)\tilde{\Gamma}_q(N+\alpha-k-n)}{[n]_q! [N-k-1-n]_q!}$$

where $\chi(k, \alpha, \beta) = k \left(\frac{\alpha + \beta + k + 2}{2} \right) - \frac{1}{2} \left[\binom{k+1}{2} + 1 \right]$, or equivalently

$$u_k^q = \frac{q^{\chi(k, \alpha, \beta)}}{[N - k - 1]_q!} \sum_{n=0}^{N-k-1} q^{\psi(\alpha, \beta, n, N)} \begin{bmatrix} N - k - 1 \\ n \end{bmatrix}_q \tilde{\Gamma}_q(\beta + k + 1 + n) \tilde{\Gamma}_q(N + \alpha - k - n).$$

For brevity we have introduced the notation: $\psi(\alpha, \beta, n, N, k) = n(\alpha + \beta + 2k + 3 + n - N)/2$.

Taking into account

$$\frac{\tilde{\Gamma}_q(\beta + k + 1 + n)}{\Gamma_q(\beta + k + 1)} = q^{-(\beta + k + n)(\beta + k + n - 1)/4} \frac{(q^{\beta + k + 1}; q)_n}{(1 - q)^n}$$

$$\frac{\tilde{\Gamma}_q(N - k - n + \alpha)}{\Gamma_q(\alpha + 1)} = q^{-(N - k - 1 - n + \alpha)(N - k - n + \alpha - 2)/4} \frac{(q^{\alpha + 1}; q)_{N - k - 1 - n}}{(1 - q)^{N - k - 1 - n}},$$

the q -moments becomes

$$u_k^q = \frac{\Gamma_q(\alpha + 1) \Gamma_q(\beta + k + 1) q^{v(\alpha, \beta, N, k)}}{(1 - q)^{N - k - 1} [N - k - 1]_q!} \sum_{n=0}^{N-k-1} q^{n(\alpha + 1)} \begin{bmatrix} N - k - 1 \\ n \end{bmatrix}_q (q^{\alpha + 1}; q)_{N - k - 1 - n} (q^{\beta + k + 1}; q)_n.$$

where

$$2(v(\alpha, \beta, N, k) + 1) = \alpha(2k - N + 1) + N(k + 1) - \binom{k}{2} - \binom{N}{2} - \binom{\alpha}{2} - \binom{\beta}{2}.$$

Using the following well-known relation (see [11, p. 25])

$$\sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q (a; q)_{n-j} (b; q)_j a^j = (ab; q)_n,$$

one gets

$$u_k^q = \frac{\Gamma_q(\alpha + 1) \Gamma_q(\beta + k + 1) q^{v(\alpha, \beta, N, k)}}{(1 - q)^{N - k - 1} [N - k - 1]_q!} (q^{\alpha + \beta + k + 2}; q)_{N - k - 1}.$$

In particular

$$u_0^q = \frac{\Gamma_q(\alpha + 1) \Gamma_q(\beta + 1) q^{v(\alpha, \beta, N, 0)}}{(1 - q)^{N - 1} [N - 1]_q!} (q^{\alpha + \beta + 2}; q)_{N - 1}.$$

On the other hand, using relations

$$\begin{aligned}\frac{\Gamma_q(\beta + k + 1)}{\Gamma_q(\beta + 1)} &= \frac{(q^{\beta+1}; q)_k}{(1 - q)^k} \\ \frac{\tilde{\Gamma}_q(N - k)}{\Gamma_q(N)} &= q^{-(N-k-1)(N-k-2)/4} \frac{(-1)^k (1 - q)^k q^{(1-N)k + \binom{k}{2}}}{(q^{1-N}; q)_k}, \\ (q^{\alpha+\beta+k+2}; q)_{N-k-1} &= \frac{(q^{\alpha+\beta+2}; q)_{N-1}}{(q^{\alpha+\beta+2}; q)_k}, \\ \Gamma_q(N) &= q^{(N-1)(N-2)/4} [N - 1]_q!,\end{aligned}$$

we obtain

$$u_k^q = u_0^q q^{k(N+\alpha) - \binom{k}{2}} (-1)^k \frac{(q^{\beta+1}; q)_k (q^{1-N}; q)_k}{(1 - q)^k (q^{\alpha+\beta+2}; q)_k}.$$

Finally, using (24) and (44) one gets the q -Stieltjes function associated with the q -Hahn moment functional

$$S_q(z) = \frac{u_0^q}{x(z)} \sum_{k \geq 0} \frac{(q^{\beta+1}; q)_k (q^{1-N}; q)_k (q; q)_k q^{k(N+\alpha-z)}}{(q^{1-z}; q)_k (q^{\alpha+\beta+2}; q)_k (q; q)_k},$$

or equivalently –in terms of hypergeometric series–

$$S_q(z) = \frac{u_0^q}{x(z)} {}_3\varphi_2 \left(\begin{matrix} q^{\beta+1}, q^{1-N}, q \\ q^{1-z}, q^{\alpha+\beta+2} \end{matrix} \middle| q; q^{N+\alpha-z} \right).$$

Now, aimed to find an equivalent relation for the above q -Stieltjes function one uses the relations

$$\tilde{\Gamma}_q(s + 1) = q^{-\binom{s}{2}/2} \frac{(q; q)_s}{(1 - q)^s},$$

and

$$\frac{\tilde{\Gamma}_q(s + \beta + 1)}{\Gamma_q(\beta + 1)} = q^{-(s+\beta)(s+\beta-1)/4} \frac{(q^{\beta+1}; q)_s}{(1 - q)^s},$$

as well as

$$\frac{\tilde{\Gamma}_q(N + \alpha - s)}{\Gamma_q(N + \alpha)} = q^{-(N+\alpha-s-1)(N+\alpha-s-2)/4} \frac{(-1)^s (1 - q)^s q^{(1-N-\alpha)s + \binom{s}{2}}}{(q^{1-N-\alpha}; q)_s}.$$

Thus, we can rewrite $\rho(s)$ as follows

$$\rho(s) = q^{\tilde{\nu}(\alpha, \beta, N)} \frac{\Gamma_q(\beta + 1) \Gamma_q(N + \alpha)}{\Gamma_q(N)} \frac{(q^{\beta+1}; q)_s (q^{1-N}; q)_s}{(q^{1-N-\alpha}; q)_s (q; q)_s},$$

where $2\tilde{\nu}(\alpha, \beta, N) = \alpha(1 - N) - \binom{\alpha}{2} - \binom{\beta}{2}$. Then, from (30) one gets

$$S_q(z) = \frac{C_q}{x(z)} \sum_{s \geq 0} \frac{(q^{\beta+1}; q)_s (q^{1-N}; q)_s (q^{-z}; q)_s q^s}{(q^{1-N-\alpha}; q)_s (q^{1-z}; q)_s (q; q)_s}, \quad N + \alpha \notin \mathbb{N},$$

where

$$C_q = q^{\tilde{v}(\alpha, \beta, N) - 1/2} \frac{\Gamma_q(\beta + 1) \Gamma_q(N + \alpha)}{\Gamma_q(N)}.$$

Therefore

$$S_q(z) = \frac{C_q}{x(z)} {}_3\varphi_2 \left(\begin{matrix} q^{\beta+1}, q^{1-N}, q^{-z} \\ q^{1-N-\alpha}, q^{1-z} \end{matrix} \middle| q; q \right). \quad (52)$$

Finally, taking into account the relation

$$u_0^q = \frac{(q^{\alpha+\beta+2}; q)_{N-1}}{(q^{\alpha+1}; q)_{N-1}} C_q,$$

and using the transformation formula (see [7, Theorem 12.4.2])

$${}_3\varphi_2 \left(\begin{matrix} q^{-n}, a, b \\ c, d \end{matrix} \middle| q; q \right) = \frac{b^n (d/b; q)_n}{(d; q)_n} {}_3\varphi_2 \left(\begin{matrix} q^{-n}, b, c/a \\ c, q^{1-n}b/d \end{matrix} \middle| q; aq/d \right),$$

taking $n = N - 1$, $a = q^{-z}$, $b = q^{\beta+1}$, $c = q^{1-z}$, and $d = q^{1-N-\alpha}$, relation (52) transforms into the following expression

$$S_q(z) = \frac{u_0^q}{x(z)} {}_3\varphi_2 \left(\begin{matrix} q^{\beta+1}, q^{1-N}, q \\ q^{1-z}, q^{\alpha+\beta+2} \end{matrix} \middle| q; q^{N+\alpha-z} \right),$$

where relation $(aq^{-n}; q)_n = (q/a; q)_n (-a)^n q^{-n(n+1)/2}$ (see [7, p. 304, (12.2.10)]) have been used.

6 Conclusions and future directions

Based on the accumulation of analytic and algebraic properties that characterize the classical discrete orthogonal polynomials -often seemed to be unrelated- the need for structure and classification of them has constantly arisen as a central question in the orthogonal polynomial theory [9, 15, 19, 20]. In this paper we address this question proving, in Theorem 1, that a sequence of q -polynomials orthogonal with respect to q -moment functional (21) is classical iff the associated Stieltjes function given in terms of q -falling factorial basis satisfies a first order non-homogeneous q -difference equation; the proof is given in a constructive way using a theoretical background based on the theory of linear functional deeply studied in [16] and [19]. We show, in Theorem 1, that the verification of the aforementioned difference equation constitutes a new characterization of a discrete orthogonal polynomials on the non-uniform lattice $x(s) = (q^s - 1)/(q - 1)$. However, more general situations demand special attention. For instance, the q -orthogonal polynomials on the lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$, ($q \in \mathbb{R}^+ \setminus \{1\}$), where the constants $c_i \in \mathbb{R}$ ($i = 1, 2, 3$) are constants independent of s , must be considered in the same fashion as here. This paper outlines the

important points and techniques to be followed in such investigations aimed to characterize those families of q -polynomials.

Finally, more general systems of q -moment functionals must be analyzed. In this direction, the q -semiclassical orthogonal polynomials could be an interesting challenge to be considered; in particular when Dirac masses are added to q -moment functional. An analogous result to those given in Theorem 1 played a crucial role in the computation of the class of the semiclassical functionals given by a perturbation via the addition of Dirac masses (see [10]).

In closing, to the best of our knowledge, there is not in the literature any explicit expression for the associated Stieltjes functions in the q -falling factorial basis given in terms of hypergeometric functions.

Acknowledgments The research of the first author was partially supported by the research grant MTM2009-12740-C03-01 of the Ministerio de Educación y Ciencia of Spain and grant CC-G08-UC3M/ESP-4516 from Comunidad Autónoma de Madrid. We thank the reviewers for offering useful suggestions for improving the paper.

References

- [1] W. Al Salam, *Characterization theorems for orthogonal polynomials*. In P. Nevai (ed.), *Orthogonal Polynomials: Theory and Practice*, Kluwer Academic Publishers, Dordrecht, 1–24, 1990.
- [2] R. Álvarez-Nodarse and J. Arvesú, *On the q -polynomials in the exponential lattice*, *Integral Transforms and Special Function* 8, 299–324, 1999.
- [3] J. Arvesú, *Quantum Algebras $SU_q(2)$ and $SU_q(1,1)$ associated with certain q -Hahn polynomials*, *Electronic Transaction on Numerical Analysis*, Volume 24, 2006, 24–44.
- [4] J. Arvesú, *Propiedades analíticas y algebraicas de polinomios con diversos modelos de ortogonalidad: q -Discretos, tipo Sobolev y semiclásicos*, Tesis Doctoral, Universidad Carlos III de Madrid, 1999.
- [5] J. Arvesú, *On some properties of q -Hahn multiple orthogonal polynomials*, *Journal of Computational and Applied Mathematics*, Vol. 233, Issue 6, 2010, 1462–1469.
- [6] M. Bradley, *Duality for finite multiple harmonic q -series*, *Discrete Mathematics*, 300, 2005, 44–56.
- [7] M.E.H. Ismail, *Classical and quantum orthogonal polynomials in one variable*. With two chapters by Walter Van Assche, *Encyclopedia of Mathematics and its Applications* 98. Cambridge: Cambridge University Press. 2005.
- [8] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

- [9] A.G. García, F. Marcellán, L. Salto, *A distributional study of discrete classical orthogonal polynomials*, Journal of Computational and Applied Mathematics, Vol. 57, 1995, 147–162.
- [10] A. Garrido, J. Arvesú, and F. Marcellán, *Modification of Linear Functionals with Dirac Masses: Class of the Modified Linear Functional*, Boletín de Matemáticas, Universidad Nacional de Colombia, 11, 2004, 32–51.
- [11] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 2004.
- [12] R. Koekoek, R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue*, Reports of the faculty of Technical Mathematics and Informatics no. 98-17, Delft, 1998 (math.CA/9602214 at arXiv.org).
- [13] F. Marcellán, A. Branquinho, and J. Petronilho, *Classical orthogonal polynomials: a functional approach*, Acta Applicandae Mathematicae, 34, 283–303, 1994.
- [14] F. Marcellán and J. Petronilho, *On the solution of some distributional differential equations: existence and characterizations of the classical moment functionals*, Integral Transforms and Special Functions, 2(3), 185–218, 1994.
- [15] F. Marcellán, and L. Salto, *Discrete semi-classical orthogonal polynomials*, Journal of Difference Equations and Applications, Vol. 4, Issue 5, 1998, 463–496.
- [16] P. Maroni, *An algebraic theory of orthogonal polynomials. Application to semiclassical orthogonal polynomials* (in French), Orthogonal polynomials and their applications (Erice, 1990), 95–130, IMACS Annals on Computing and Applied Mathematics, 9, Baltzer, Basel, 1991.
- [17] P. Maroni, *Variations around classical orthogonal polynomials. Connected problems*, Journal of Computational and Applied Mathematics, 48, 133–155, 1993.
- [18] P. Maroni, *Fonctions eulériennes. Polynômes orthogonaux classiques*, Techniques de l'Ingénieur, traité Généralités (Sciences Fondamentales) A 154, 1–30, 1994.
- [19] J.C. Medem, R. Álvarez-Nodarse, and F. Marcellán, *On the q -polynomials: a distributional study*, Journal of Computational and Applied Mathematics, 135, 2001, 157–196.
- [20] A.F. Nikiforov, S.K. Suslov and V.B. Uvarov, *Classical Orthogonal polynomials of a Discrete Variable*, Springer Series in Computational Physics, Springer-Verlag, Berlin, 1991.

- [21] A.F. Nikiforov, V.B. Uvarov, *Special Functions of Mathematical Physics*, Birkhäuser Verlag, Basel, 1988.
- [22] E.M. Nikishin, V.N. Sorokin, *Rational Approximations and Orthogonality*, Translations of Mathematical Monographs, vol. 92, Amer. Math. Soc., Providence, RI, 1991.
- [23] T.J. Stieltjes, *Oeuvres complètes/Collected papers*, Vol. 1 and 2 (English and French Edition) Edited by G. van Dijk, Springer, 1993.
- [24] S.K. Suslov, *The theory of difference analogues of special functions of hypergeometric type*, Russian Mathematical Surveys 44:2, 1989, 227–278.
- [25] B. Simon, *The Classical Moment Problem as a Self-Adjoint Finite Difference Operator*, Advances in Mathematics, 137, 1998, 82–203.